16.2: Vector Fields

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Last class

Definition

If $F(x, y, z)$ is defined on a curve C given parametrically by $\vec{\mathbf{r}}(t) = f(t)\vec{\mathbf{i}} + g(t)\vec{\mathbf{j}} + h(t)\vec{\mathbf{k}}$, $a \leq t \leq b$, then the line integral of F over C is

$$\int_C F(x, y, z)ds = \lim_{n\to\infty} \sum_{k=1}^{n} F(x_k, y_k, z_k)\Delta s_k$$

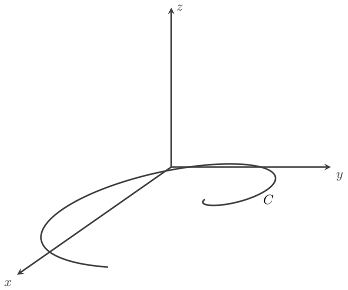
Evaluating

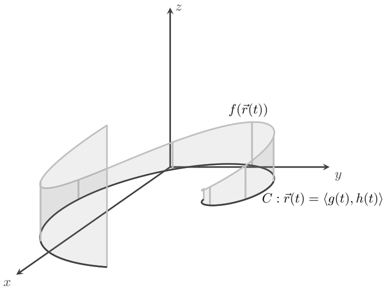
To turn a line integral into an integral with respect to t, we replace $F(x, y, z)$ by $F(f(t), g(t), h(t))$, and ds by $\|\vec{\mathbf{v}}(t)\|dt$. So

$$\int_C F(x, y, z)ds = \int_{t=a}^{t=b} F(f(t), g(t), h(t))\|\vec{\mathbf{v}}(t)\|dt.$$

Geometric representation of a line integral

In the case that the curve C is 2-dimensional (i.e., restricted to the xy-plane), and the function is a 2-variable function $f(x, y)$, then the line integral of $f(x, y)$ over C is the area between the function and the curve.





Vector Fields

Definition

A vector field is a function

$$\overrightarrow{\mathbf{F}} = M(x, y, z)\vec{\mathbf{i}} + N(x, y, z)\vec{\mathbf{j}} + P(x, y, z)\vec{\mathbf{k}}.$$

(i.e., a function that gives you a vector for every point in space).

A vector field could represent the flow of a liquid through a pipe, or the force of gravity, or any application where a force is involved.

Line integral of a vector field

We can integrate a vector field along a curve. This quantity keeps track of the work a vector field does on a particle traveling along the curve.

Definition

Let $\overrightarrow{\mathbf{F}}$ be a vector field with continous components defined a long a smooth curve C parametrized by $\vec{\mathbf{r}}(t), a \leq t \leq b$. Then the line integral of $\overrightarrow{\mathbf{F}}$ along C is

$$\int_C \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{T}} ds = \int_C (\overrightarrow{\mathbf{F}} \cdot \frac{d\vec{\mathbf{r}}}{ds}) ds = \int_C \overrightarrow{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_a^b (\overrightarrow{\mathbf{F}} \cdot \frac{d\vec{\mathbf{r}}}{dt}) dt.$$

Evaluating

If we are given $\overrightarrow{\mathbf{F}} = M\vec{\mathbf{i}} + N\vec{\mathbf{j}} + P\vec{\mathbf{k}}$ and $C : \vec{\mathbf{r}}(t) = g(t)\vec{\mathbf{i}} + h(t)\vec{\mathbf{j}} + k(t)\vec{\mathbf{k}}$, then we follow the procedure below:

1. Write $\overrightarrow{\mathbf{F}}$ in terms of C: $\overrightarrow{\mathbf{F}}(\vec{\mathbf{r}}(t))$.
2. Find $\frac{d\vec{\mathbf{r}}}{dt}$.
3. Evaluate with respect to t as

$$\int_C \overrightarrow{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_a^b \left(\overrightarrow{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \frac{d\vec{\mathbf{r}}}{dt}\right) dt$$

Example

Example

Let $\overrightarrow{\mathbf{F}}(x,y,z) = z\vec{\mathbf{i}} + xy\vec{\mathbf{j}} - y^2\vec{\mathbf{k}}$ *be a vector field. Find the integral of* $\overrightarrow{\mathbf{F}}$ *along* C *given by* $\vec{\mathbf{r}}(t) = t^2\vec{\mathbf{i}} + t\vec{\mathbf{j}} + \sqrt{t}\vec{\mathbf{k}}$, $0 \le t \le 1$.

We have $\overrightarrow{\mathbf{F}}(\vec{\mathbf{r}}(t)) = \sqrt{t}\vec{\mathbf{i}} + t^3\vec{\mathbf{j}} - t^2\vec{\mathbf{k}}$, and $\frac{d\vec{\mathbf{r}}}{dt} = 2t\vec{\mathbf{i}} + \vec{\mathbf{j}} + \frac{1}{2\sqrt{t}}\vec{\mathbf{k}}$. So

$$\int_C \overrightarrow{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_0^1 \left(\overrightarrow{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \frac{d\vec{\mathbf{r}}}{dt}\right) dt = \int_0^1 (2t^{3/2} + t^3 - \frac{1}{2}t^{3/2})dt$$

$$= \frac{4}{5}t^{5/2} + \frac{t^4}{4} - \frac{1}{5}t^{5/2}\Big]_0^1 = \frac{17}{20}$$